PHYS 301 Assign. #3 Sol'ns Oct. 26, 2024

1. $\nabla^2 V$ is cylindrical coords. Assume $V = V(s, \phi)$

(no z-dependence)

$\therefore V = S(s)\,\Phi(\phi)$

$$\therefore \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$\therefore \frac{\Phi}{\cancel{s}}\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{S}{s^{\cancel{2}}}\frac{d^2\Phi}{d\phi^2} = 0$$

mult. by $\frac{s}{S\Phi}$

$$\therefore \frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\Rightarrow s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = k^2 S \qquad \frac{d^2\Phi}{d\phi^2} = -k^2\Phi$$

1 2

Sol'n to ②

$$\Phi = A\sin k\phi + B\cos k\phi$$

$$\frac{d\Phi}{d\phi} = kA\cos k\phi - kB\sin k\phi$$

$$\frac{d^2\Phi}{d\phi^2} = -k^2 A\sin k\phi - k^2 B\cos k\phi$$

$$= -k^2\Phi \checkmark$$

require

$\Phi(\phi+2\pi) = \Phi(\phi)$

$\therefore \sin[k(\phi+2\pi)]$

$= \sin[k\phi + 2\pi k]$

$= \sin(k\phi)$

$\therefore k$ integer.

$k = 0, 1, 2, \ldots$

Sol'n to ①

$$\frac{d}{ds}\left(s\frac{dS}{ds}\right) = \frac{k^2 S}{s}$$

Try $S = Cs^n$

$$\therefore \frac{d}{ds}\left(s\, nCs^{n-1}\right) = nC\frac{d}{ds}\left(s^n\right) = n^2 C s^{n-1} = \frac{k^2 S}{s}$$

$$\therefore n^2 \underbrace{Cs^n}_{S} = k^2 S$$

$\therefore n^2 = k^2$

$\therefore n = \pm k.$

$\therefore \; S = C s^{k} + D s^{-k}$

<u>Note:</u> If $k=0$

$$\frac{d}{ds}\left(s\frac{dS}{ds}\right)=0$$

$\therefore \; \dfrac{dS}{ds} = \dfrac{E}{s} \qquad \therefore \; S = E\ln s + F$

$\dfrac{d^2\Phi}{d\phi^2}=0 \qquad \therefore \; \Phi = G\phi + H$

This does <u>not</u> satisfy

$\Phi(\phi+2\pi) = \Phi(\phi)$

$\therefore G=0$

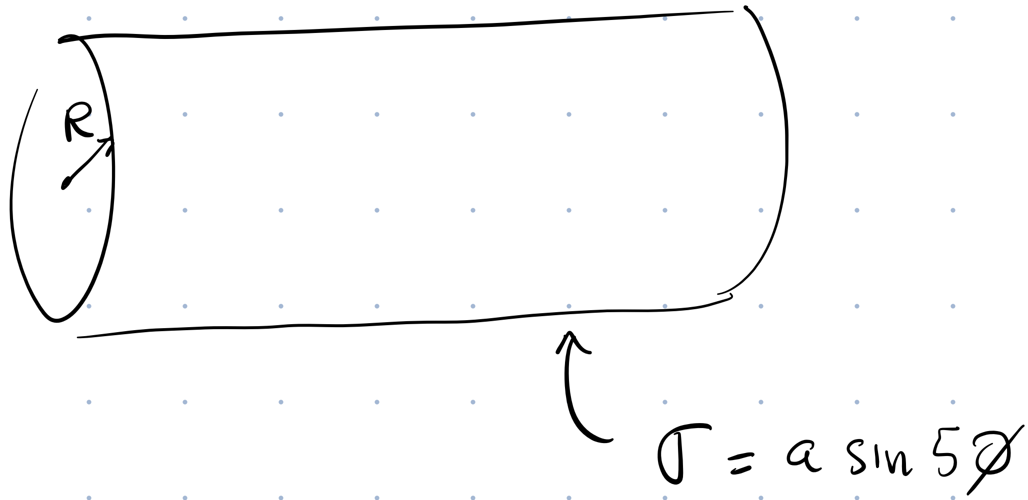
& $\Phi = H$.

$$\therefore\ V(s,\phi) = \underbrace{\left(F + E\ln s\right)}_{k=0\ \text{case}} + \sum_{k=1}^{\infty}\left(C_k s^k + D_k e^{-k}\right)\cdot\left(A_k \sin k\phi + B_k \cos k\phi\right)$$

Re-label constants to follow the notation used by Griffiths:

$$\boxed{V(s,\phi) = a_0 + b_0 \ln s + \sum_{k=1}^{\infty}\left[s^k\left(a_k \cos k\phi + b_k \sin k\phi\right) + s^{-k}\left(c_k \cos k\phi + d_k \sin k\phi\right)\right]}$$

2.



Inside case: $s < R$

Boundary conditions:

(i) V is finite when $s = 0$

$\ln s$ & s^{-k} diverge as $s \to 0$

$\therefore \; b_0 = 0$

$c_k = 0$

$d_k = 0$

$$\boxed{\therefore \; V_{in}(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^k \left(a_k \cos k\phi + b_k \sin k\phi \right)}$$

outside case: $s > R$

Boundary conditions:

(i) $V \to 0$ as $s \to \infty$

$\ln s \to \infty$ & $s^k \to \infty$ as $s \to \infty$

$\therefore\ b_0 = 0,\quad a_k = 0,\quad b_k = 0$

$$\boxed{V_{out}(s,\phi) = a_0' + \sum_{k=1}^{\infty} s^{-k}\left(c_k \cos k\phi + d_k \sin k\phi\right)}$$

possibly a different constant than in the $s < R$ case.

Know the b.c. for a surface charge σ is

$$\underbrace{E^{\perp}_{outside}}_{-\frac{\partial V_{out}}{\partial s}} - \underbrace{E^{\perp}_{inside}}_{-\frac{\partial V_{in}}{\partial s}} = \frac{\sigma}{\varepsilon_0}$$

$$\therefore \left.\frac{\partial V_{out}}{\partial s}\right|_{s=R} - \left.\frac{\partial V_{in}}{\partial s}\right|_{s=R} = -\frac{a}{\varepsilon_0} \sin 5\phi$$

$$\therefore \cancel{-} \sum_{k=1}^{\infty} k R^{-(k+1)} \left(c_k \cos k\phi + d_k \sin k\phi \right)$$

$$\cancel{-} \sum_{k=1}^{\infty} k R^{k-1} \left(a_k \cos k\phi + b_k \sin k\phi \right)$$

$$= \cancel{-} \frac{a}{\varepsilon_0} \sin 5\phi$$

$$\therefore \sum_{k=1}^{\infty} k \left[R^{-(k+1)} \left(c_k \cos k\phi + d_k \sin k\phi \right) + R^{k-1} \left(a_k \cos k\phi + b_k \sin k\phi \right) \right]$$

$$= \frac{a}{\varepsilon_0} \sin 5\phi$$

only $k=5$ term is relevent.

$$5\left[R^{-6}\left(c_5\cos 5\phi + d_5\sin 5\phi\right) + R^{4}\left(a_5\cos 5\phi + b_5\sin 5\phi\right)\right] = \frac{a}{\varepsilon_0}\sin 5\phi$$

$$\therefore \left(\frac{c_5}{R^6} + a_5R^4\right)\cos 5\phi + \left(\frac{d_5}{R^6} + b_5R^4\right)\sin 5\phi = \frac{a}{5\varepsilon_0}\sin 5\phi$$

$$\boxed{\therefore c_5 = -a_5R^{10}} \quad \circledast$$

$$\therefore \frac{d_5}{R^6} + b_5R^4 = \frac{a}{5\varepsilon_0} \Rightarrow \boxed{a = 5\varepsilon_0\left(\frac{d_5}{R^6} + b_5R^4\right)}$$

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Also require $V_{in} = V_{out}$ @ $s = R$

$$\therefore\ a_0 + R^5\left(a_5 \cos 5\phi + b_5 \sin 5\phi\right)$$

$$= a_0' + \frac{1}{R^5}\left(c_5 \cos 5\phi + d_5 \sin 5\phi\right)$$

$$\therefore\ R^5 a_5 = \frac{1}{R^5} c_5$$

$$\boxed{\therefore\ c_5 = a_5 R^{10}}$$

✗ was $c_5 = -a_5 R^{10}$

$\Rightarrow$ must be the case that $\underline{\underline{c_5 = a_5 = 0}}$

$$\boxed{R^5 b_5 = \frac{1}{R^5} d_5} \Rightarrow d_5 = R^{10} b_5$$

sub into (4)

$$a = 5\varepsilon_0 \left(\frac{R^{10} b_5}{R^6} + b_5 R^4 \right)$$

$$a = 10\,\varepsilon_0 R^4 b_5 \qquad \therefore \boxed{b_5 = \frac{a}{10\varepsilon_0 R^4}}$$

$$\boxed{d_5 = \frac{a R^6}{10\,\varepsilon_0}}$$

Also require $a_0 = a_0'$.

Since $\vec{E} = -\vec{\nabla} V$, can choose whatever we want for the a_0 const w/o affecting $\vec{E}$. Take $a_0 = a_0' = 0$

$$\boxed{\therefore V_{in}(r, \phi) = \frac{a s^5}{10\varepsilon_0 R^4} \sin 5\phi}$$

$$\boxed{V_{out}(r, \phi) = \frac{a R^6}{10 \varepsilon_0 s^5} \sin 5\phi}$$